GEOPHYSICS, VOL. 74, NO. 6 (NOVEMBER-DECEMBER 2009); P. WCD1–WCD13, 1 FIG.
10.1190/1.3256286

Clarifying the underlying and fundamental meaning of the approximate linear inversion of seismic data

Arthur B. Weglein[1], Haiyan Zhang[2], Adriana C. Ramírez[3], Fang Liu[1], and Jose Eduardo M. Lira[4]

ABSTRACT

Linear inversion is defined as the linear approximation of a direct-inverse solution. This definition leads to data requirements and specific direct-inverse algorithms, which differ with all current linear and nonlinear approaches, and is immediately relevant for target identification and inversion in an elastic earth. Common practice typically starts with a direct forward or modeling expression and seeks to solve a forward equation in an inverse sense. Attempting to solve a direct forward problem in an inverse sense is not the same as solving an inverse problem directly. Distinctions include differences in algorithms, in the need for a priori information, and in data requirements. The simplest and most accessible examples are the direct-inversion tasks, derived from the inverse scattering series (ISS), for the removal of free-surface and internal multiples. The ISS multiple-removal algorithms require no subsurface information, and they are independent of earth model type. A direct forward method solved in an inverse sense, for modeling and subtracting multiples, would require accurate knowledge of every detail of the subsurface the multiple has experienced. In addition, it requires a different modeling and subtraction algorithm for each different earth-model type. The ISS methods for direct removal of multiples are not a forward problem solved in an inverse sense. Similarly, the direct elastic inversion provided by the ISS is not a modeling formula for PP data solved in an inverse sense. Direct elastic inversion calls for PP, PS, SS, … data, for direct linear and nonlinear estimates of changes in mechanical properties. In practice, a judicious combination of direct and indirect methods are called upon for effective field data application.

INTRODUCTION

We begin with a set of definitions and a discussion of terms and concepts used here. We illustrate how these terms are used within a context of current and conventional seismic processing. That assists identifying how the contribution, message, and algorithms of this paper depart from and add to the current understanding and advancement of seismic theory and practice.

In the next section, we define forward and inverse processes and problems, define direct and indirect solutions, describe modeling as a direct forward procedure, and introduce and define intrinsic and circumstantial nonlinearity.

DEFINITIONS, CENTRAL ISSUES AND GOALS OF DIRECT NONLINEAR INVERSION, AND DISTINGUISHING INDIRECT FULL-WAVEFORM MODEL-MATCHING FROM DIRECT INVERSION

Forward and inverse problems

A forward problem inputs the medium properties and the source character and outputs the wavefield everywhere inside and outside the medium of interest. The inverse problem inputs measurements of the wavefield outside the medium of interest and the source character. It outputs processing goals that include locating structure/reflectors at their correct spatial location and identifying the changes in the earth's mechanical properties across the imaged reflectors. We adopt the inclusive definition of inversion, which accommodates (1) the determination of subsurface properties, e.g., structure and medium properties, and (2) intermediate inversion goals associated with processing tasks (like multiple removal) that facilitate subsequent determination of structure and medium properties.

Manuscript received by the Editor 6 February 2009; revised manuscript received 9 July 2009; published online 15 December 2009.
[1]University of Houston, Houston, Texas, U.S.A. E-mail: aweglein@uh.edu; liufang23@hotmail.com.
[2]Formerly University of Houston, Houston, Texas, U.S.A.; presently ConocoPhillips, Houston, Texas, U.S.A. E-mail: Haiyan.Zhang@conocophillips.com.
[3]Formerly University of Houston, Houston, Texas, U.S.A.; presently WesternGeco, Houston, Texas, U.S.A. E-mail: acramirez@slb.com.
[4]PETROBRAS Research and Development Center, Rio de Janeiro, Brazil. E-mail: jelira@petrobras.com.br.

Direct and indirect methods

Methods for achieving these forward and inverse goals are classified as direct or indirect. Modeling methods are typically direct: they input medium properties and output the wavefield directly. Inverse methods that input seismic recorded data and output medium properties, or other seismic-processing objectives, straight away and directly are direct inverse methods. Indirect seismic inversion or processing methods do not output medium properties or seismic processing objectives directly. Instead, they seek and search, locally and globally, and consider possible candidates that emulate a characteristic, property or invariance that a direct solution would automatically satisfy (e.g., Tarantola, 1990; Stoffa and Sen, 1991; Pratt, 1999; Sirgue and Pratt, 2004; Landa et al., 2006; Vigh and Starr, 2008, and references therein). Through the satisfaction of that property, indirect methods seek a solution.

Direct modeling methods

There are many direct methods that model and generate seismic data. Modeling methods include (1) finite difference, (2) finite element, (3) reflectivity, (4) Cagniard-De Hoop, (5) lattice Boltzmann, and (6) the forward-scattering series. Another example of modeling is given by the Zoeppritz equations. The Zoeppritz equations model elastic plane-wave reflection and transmission coefficients, and provide closed-form expressions that input changes in mechanical properties across a horizontal boundary, the incident plane-wave angle, and the type of incident wave to predict, directly and nonlinearly, the various elastic reflection and transmission coefficients of waves that are generated by the incident plane wave and that emanate from the boundary. All of these modeling methods input medium properties and directly output the wavefield. They are direct modeling methods.

Defining intrinsic and circumstantial nonlinearity and their roles in direct inversion theory

Certain forward and inverse processes commonly are recognized as inherently nonlinear. The most well known are the Zoeppritz relationships between the changes in mechanical properties across a horizontal interface between two elastic half-spaces and the reflection and transmission coefficients given by the Zoeppritz equations. Zoeppritz is a forward direct and closed-form nonlinear relationship, and it is the archetypical intrinsic or innate nonlinearity. Intrinsic (or innate) means that only detailed accurate information everywhere in the subsurface can avoid that nonlinearity. The forward nonlinearity implies an inverse nonlinearity. From an inverse point of view, the only way to avoid that Zoeppritz type of nonlinearity is to know the entire subsurface. If one is interested in determining the mechanical properties in any region of the subsurface that initially is unknown, then one is facing intrinsic nonlinearity. If we assume, e.g., complete knowledge of all medium properties (not only velocity) down to a given reflector, and what is beneath that reflector is unknown, then we are facing the nonlinear inverse of inverting the nonlinear forward Zoeppritz equations and/or their multidimensional generalization for property changes across that reflector.

Processes that are sometimes linear or nonlinear, circumstantial nonlinearity, removing multiples, and depth imaging primaries

There are other forward and inverse processes that are commonly, reasonably, and correctly considered as linear. For example, (1) depth imaging for structure with an accurate velocity model and (2) modeling and subtracting water-bottom multiples are linear methods for migration and multiple removal, respectively. Each case requires a priori information but not for the entire subsurface — only enough a priori information to achieve the stated goal. In circumstances where relevant and accurate a priori information is unavailable or inadequate to achieve the two above-mentioned goals (directly depth imaging with an accurate velocity model and modeling and subtracting multiples), the inverse scattering series (ISS) offers the opportunity to achieve each of these two goals (which are linearly achievable with a priori information) directly and nonlinearly in terms of the data and without a priori information. We define that type of nonlinearity as circumstantial nonlinearity.

Third type of nonlinearity: the combination of intrinsic and circumstantial

There is a third kind of nonlinearity that is a combination of the intrinsic and circumstantial types. The third kind of nonlinearity can take place in, e.g., a situation where the goal is to determine the location and changes in mechanical properties across a specific reflector and there is an unknown overburden above that reflector. The latter goal also is within the promise and purview of the inverse scattering series. It is directly achievable in terms of nonlinear relationships of the data, without knowing, needing, or determining overburden information.

The ISS is the only method that can directly invert and address either the intrinsic or the circumstantial nonlinearity, when they occur separately, let alone accommodate this nonlinearity when they occur simultaneously, i.e., together and in combination. An example of a combined (type-three nonlinearity) is the direct target identification beneath an unknown overburden. Target identification is intrinsically nonlinear by itself and the unknown overburden adds circumstantial nonlinearity to the mix.

The ISS is the only multidimensional direct inversion for acoustic or elastic media. However, it took that general ISS machinery to provide the first direct inverse solution to the simplest and archetypical single-interface intrinsic nonlinear forward problem, defined by the forward Zoeppritz equations. The ISS provides an order-by-order (in terms of data) solution for inverting that type of plane-wave reflection data to determine the changes in mechanical properties across a specific reflector. What reflection data are required as input to allow this first direct solution to provide the changes in mechanical properties across that single interface?

The message from the only direct inverse method is that PP data are fundamentally insufficient for direct linear or nonlinear inversion, and that all components PP, PS, SS, …are required before one gets started. The direct order-by-order solution for any one or all changes in mechanical properties across that single reflector, explicitly call upon all those independent data components. That message is itself at variance with the extensive published literature on target identification, elastic parameter estimation, amplitude variation with offset (AVO), full-waveform inversion, iterative linear inversion, global and local search engines, optimization schemes, model matching, common-image-gather flatness, and optimal trajectory stacking, i.e., "path integral" — essentially all current seismic practice and theory. The only direct inversion method has a definitive statement on how (for the first time) to directly compute linear and higher-order terms for the changes in mechanical properties.

Perhaps it should come as no surprise that this direct solution has a view that is different and at odds with every manner of indirect thinking and algorithm. The divergence between direct and indirect thinking and algorithms is immediate, significant, and substantive for the case of elastic target identification and inversion, and methods that depend on those results. In the elastic case, direct inversion communicates unambiguously and clearly that indirect inversion methods today are using wrong and fundamentally insufficient data. Indirect methods on their own have no way of recognizing it.

Given the choice between the direct and definitive solution of the equation $x + 1 = 2$, found by subtracting 1 from both sides of the equation, and indirect searching for numbers that make some weighted integrated measure of the difference between $x + 1$ and 2 small, which one would we trust, and have confidence in, as a methodology and framework?

When is an inverse problem (or processing goal) the direct forward problem run backward?

In cases where a specific inverse task is directly and exactly achievable in a linear manner with adequate a priori information (for example, determining the spatial location of reflectors given an accurate velocity model, or modeling and removing water-bottom multiples), then modeling with a priori information is essential to achieve the inverse task. It is only under that type of circumstance that a link between the direct modeling and direct inverse exists. These two linear-inverse-task examples are derivable starting from the wave equation, wave propagation and imaging methods, or wave modeling and subtraction methods, respectively.

When is an inverse problem (or processing goal) not the direct forward problem solved in an inverse sense?

For any nonlinear inverse problem (whether intrinsic, circumstantial or a combination) the direct forward problem solved in an inverse sense and the direct solution of the inverse problem are not equal. Removing multiples without a priori information is achievable with a direct-inversion ISS subseries, but removing multiples with direct forward modeling and subtraction requires all a priori information that relates to the subsurface experience of the multiples. Solving a forward problem in an inverse sense for changes in mechanical properties would imply PP data are necessary and sufficient, but solving the direct elastic inverse problem for changes in mechanical properties requires all PP, PS, SS, and SP components as necessary and sufficient data for linear and higher-order direct inversion estimates.

An amazing and fortuitous property concerning how the ISS addresses circumstantial nonlinearity

An additional intriguing and amazing fact concerns a shared property of all inverse scattering subseries that address circumstantial nonlinearity. ISS subseries that address any circumstantial nonlinearity always determine first if their services are called for in any given data set, and if so, where within the data, and to what extent and degree their assistance is needed. As a reminder, a circumstantial nonlinearity represents a lack of available or adequate a priori information. It becomes a linear problem when adequate a priori information is available. Depth imaging is a linear problem with a weighted sum over data (i.e., linear in data) given the availability of an adequate velocity model. The first term in the imaging subseries is the conventional linear-imaging result, which would be the depth-imaging answer if a priori velocity information and a concomitant linear-imaging algorithm were both adequate. The second term in the imaging subseries (the first beyond linear) determines inside its integrand if the a priori information and concomitant images are adequate. If so, the integrand is zero and it signals to the rest of the imaging subseries that there is nothing for them to do with this particular data set. Subseries that address circumstantial nonlinearity (whether for depth imaging or removing internal multiples) decide first from the data if there is a need for their services. They go into action only if they decide they are needed. That is called *purposeful perturbation* theory — a consequence of the intelligence and purposefulness of direct inversion. With this as a background and motivation, we review the ISS briefly, providing the basis and justification for the statements made in this introduction, and move toward our specific message and goal.

The content that follows will be in two major sections. First, we review the inverse scattering series and provide the framework for the issues we address here. Then, we describe our subsequent thinking and issues that relate to (1) the distinction between solving a direct forward algorithm in an inverse sense and a direct inverse, (2) how the direct inverse solution stands alone in providing the clarity of explicit solutions and the data those direct inverse solutions demand, and (3) the meaning of linear inverse as linear approximation in the data and that linear inverse actually corresponds to the first and linear estimate of a nonapproximate and nonlinear direct inverse solution.

Next, we describe that thinking as it actually occurred and evolved in our research and discussions. We also describe the apparent obstacles in logical consistency and their resolution on the road that the authors traveled which culminated in our message here. We recognize that this section is not typical for scientific reporting, but the reader might appreciate and hopefully benefit from our steps along the path and from our deliberations that eventually arrived at that thinking, rather than a simple delivery of the conclusions.

INVERSE SCATTERING SERIES: THE ONLY DIRECT AND NONLINEAR INVERSION FOR A MULTIDIMENSIONAL SUBSURFACE

As noted above, there are many direct forward or modeling methods. However, the ISS is the only direct inversion method for a multidimensional acoustic, elastic, or anelastic earth.

The ISS can accommodate both the intrinsic and the circumstantial nonlinearity, separately and in combination. The nonlinearities are accommodated directly in terms of data, without the need in principle or practice to determine or estimate actual properties that govern wave propagation in the subsurface. The inverse series is the only inverse method with the capability of directly addressing and inverting either type of nonlinearity. It is also unique in its communication that, starting from one single set of ISS equations, (1) all processing goals and objectives can be achieved in the same essential template and manner, with distinct isolated-task inverse scattering subseries for each processing goal; and (2) with the same use of the amplitude and phase of seismic data directly and without subsurface information (as free-surface multiples are removed). These qualities and properties are unique to the ISS.

The ISS (Weglein et al., 2003, Weglein et al., 1997 and references therein) has the following characteristics:

1) It contains the capability to achieve all major processing objectives: free-surface multiple removal, internal multiple removal, depth imaging, Q compensation, and direct nonlinear target identification. All objectives are achievable directly in terms of measured data without a need to know, determine, or even to estimate approximately any information related to subsurface properties that govern the propagation of waves in the actual subsurface. In contrast, this information is required by conventional linear migration and migration-inversion methods to locate and identify targets.
2) Within the inverse scattering series, distinct direct algorithms input the data and output each of the processing objectives listed in item (1) through the introduction of the isolated-task subseries concept.
3) Among the tasks listed above: the first and second are each achievable by a distinct earth-model-type independent algorithm, without a single line of code that changes for an acoustic, elastic, heterogeneous, anisotropic, or anelastic earth.
4) For the removal of free-surface and internal multiples, the inverse series performs those distinct inverse tasks without dividing any quantity or inverting any matrix. These two multiple-removal tasks involve only multiplying data times data, which accounts for their robustness and stability.

For tasks that go beyond multiple removal (e.g., depth imaging, nonlinear direct AVO, and Q compensation) the inverse step is always the same. In the marine case, this step is only in terms of water-speed whole-space Green's functions; it is provided by a single water-speed FK-Stolt migration, and involves a single, unchanged, analytic algebraic division in the Fourier domain for each term in the inverse series. No need exists for a generalized inverse, model matching, indirect approaches or proxies for subsurface information, searches (local or global), downward continuation and stripping, or background updating schemes with their well-documented issues and pitfalls of low-frequency data demands and often inadequate earth-model types.

SCATTERING THEORY, THE FORWARD OR MODELING SERIES, AND THE INVERSE SCATTERING SERIES

Scattering theory is a perturbation theory. It provides the exact manner in which alterations (perturbations) in any and/or all medium properties relate to the concomitant change (perturbation) in the wavefield that experiences the altered (perturbed) medium. We map the language of scattering theory to the purposes of seismic exploration by considering the actual earth properties as consisting of a reference medium (chosen by us) plus a perturbation of the reference medium, where the combination of reference and perturbation correspond to the actual subsurface. Scattering theory then relates the perturbation (the difference between the reference and actual medium properties) to the scattered wavefield (the difference between the reference and actual medium wavefields). We begin with the basic wave equations governing wave propagation in the actual medium,

$$LG=\delta, \tag{1}$$

and in the reference medium,

$$L_0G_0=\delta, \tag{2}$$

where L and L_0 are differential operators that describe wave propagation in the actual and reference media, respectively, and G and G_0 are the corresponding Green's operators. The δ on the right side of both equations is a Dirac delta operator and represents an impulsive source. Throughout this paper, quantities with subscript "0" are for the reference medium, and those without the subscript are for the actual medium.

Following closely Weglein et al. (1997), Weglein et al. (2002), and Weglein et al. (2003), we define the perturbation $V=L_0-L$. The Lippmann-Schwinger equation

$$G=G_0+G_0VG \tag{3}$$

is an operator identity relating G, G_0, and V (see, e.g., Taylor, 1972). Iterating this equation back into itself generates the forward-scattering series

$$G=G_0+G_0VG_0+G_0VG_0VG_0+\cdots. \tag{4}$$

Then the scattered field $\psi_s \equiv G-G_0$ can be written as

$$\psi_s=G_0VG_0+G_0VG_0VG_0+\cdots=(\psi_s)_1+(\psi_s)_2+\cdots, \tag{5}$$

where $(\psi_s)_n$ is the portion of ψ_s that is n^{th} order in V.

Modeling methods, such as finite differences and finite element, generate the wavefield directly with input in terms of actual medium properties. Forward-scattering theory also models data with the actual medium properties but being a perturbation theory, the prescribed medium properties are separated into L_0 and V. The actual wavefield G is provided in terms of L, where $L=L_0-V$, L_0 enters through G_0, and V enters as V. The expansion of $G-G_0$ in orders of V is unique and is a generalized Taylor (really geometric) series with first term $a=G_0$ and the rate $r=VG_0$. This forward-scattering or forward-modeling equation communicates that any change in medium properties between L_0 and L, characterized by perturbation operator V, will lead to a change in the wavefield that is always related nonlinearly to V. Any change in medium properties at a single point, throughout a region, on a surface, or everywhere in space, or a change of medium properties of whatever magnitude at any single point will instigate this nonlinear response.

This forward-scattering relationship is the complete and multidimensional extension and generalization of the Zoeppritz relations where any change in any mechanical property across a single reflector produces reflection coefficients that are related nonlinearly to (and generated by) the change in mechanical property. The forward nonlinear relationship between the scattered field $G-G_0$ and the medium perturbation V implies a nonlinear relationship in the opposite direction of V nonlinearly related to the scattered wavefield. The latter supposition is supported by the simple geometric series analog for $G-G_0=S=ar/(1-r)$ and then $r=S/(S+a)$ and a series in S/a. The inversion problem relates data (or measured values of $G-G_0$) to V and leads to the ISS. Terms in the inverse series are an expansion of V in orders of the measured data and a generalization of an inverse geometric series — and each term in that nonlinear expansion is unique. Now, we will show that substituting this inverse series form into the forward series provides an equation for each order of V's expansion V_n that provides a unique and exact solution for that order of contribution to V. The measured values of ψ_s are the data D, where

$$D = (\psi_s)_{\mathrm{ms}}, \tag{6}$$

in which ms represents "on the measurement surface." In the ISS, expanding V as a series in orders of D,

$$V = V_1 + V_2 + V_3 + \cdots, \tag{7}$$

then substituting equation 7 into equation 5 and evaluating equation 5 on the measurement surface yields

$$D = [G_0(V_1 + V_2 + \cdots)G_0]_{ms} + [G_0(V_1 + V_2 + \cdots)G_0(V_1 + V_2 + \cdots)G_0]_{ms} + \cdots. \tag{8}$$

Setting terms that have equal order in the data equal leads to the equations that determine $V_1, V_2, \ldots$ directly from D and G_0:

$$D = [G_0 V_1 G_0]_{ms}, \tag{9}$$

$$0 = [G_0 V_2 G_0]_{ms} + [G_0 V_1 G_0 V_1 G_0]_{ms}, \tag{10}$$

and

$$0 = [G_0 V_3 G_0]_{ms} + [G_0 V_1 G_0 V_2 G_0]_{ms} + [G_0 V_2 G_0 V_1 G_0]_{ms} + [G_0 V_1 G_0 V_1 G_0 V_1 G_0]_{ms}. \tag{11}$$

Equations 9–11 permit the sequential calculation of $V_1, V_2, \ldots$, and, hence, achieve full inversion for V (see equation 7) from the recorded data D and the reference wavefield (i.e., the Green's operator of the reference medium) G_0. Therefore, the ISS is a multidimensional inversion procedure that directly determines physical properties using only reflection data and reference medium information. The reference medium is often chosen as water in the marine case.

If the subsurface medium properties V can be determined directly from data and water speed, then all intermediate steps toward that goal (e.g., removing free-surface and internal multiples, depth imaging, nonlinear direct AVO, and Q compensation) each can be achieved directly and nonlinearly in terms of data and a single, unchanged reference medium of water. Earlier in this paper, we defined different types of nonlinearity: (1) intrinsic, (2) circumstantial, and (3) the combination. The ISS, in producing changes in medium properties V from reflection data $G - G_0$, is directly and uniquely providing the order-by-order solution to the intrinsic nonlinearity, which we associate with inverting the Zoeppritz equations and multidimensional target-identification generalizations. Furthermore, because all objectives and tasks associated with inversion are achieved using the ISS directly in terms of data and water speed without a priori information, then issues involving circumstantial nonlinearity also are contained as distinct task-specific subseries of the ISS. The ISS is direct and nonlinear; it is the most comprehensive data-driven machine.

For our purposes here, the absolutely critical point to recognize at this juncture is that the equations for $V_1, V_2, \ldots$ are exact equations for $V_1, V_2, \ldots$, where $V_1, V_2, \ldots$ are linear and quadratic estimates for V, respectively…but the equations for $V_1, V_2, \ldots$ are the exact equations for the latter quantities. That the equations for $V_1, V_2, \ldots$ are each exact for those quantities is a rigorous mathematical result derived from the theorem that equal orders in a parameter (data) are equal on both sides of an equation.

Below, we present the progression of thinking that led to the message and conclusions of this paper, starting with the simpler acoustic case as a warm-up and training exercise, and progressing to the elastic world where the situation is more complicated and the consequences are significant and substantive.

ACOUSTIC CASE

We begin to examine issues that relate to necessary and sufficient data requirements for direct linear and nonlinear inversion algorithms in the relatively simple acoustic world. In this section, we will consider a 1D acoustic two-parameter earth model (e.g., bulk modulus and density or velocity and density). We start with the 3D acoustic wave equations in the actual and reference media:

$$\left[\frac{\omega^2}{K(\mathbf{r})} + \nabla\cdot\frac{1}{\rho(\mathbf{r})}\nabla\right]G(\mathbf{r},\mathbf{r}_s;\omega) = \delta(\mathbf{r}-\mathbf{r}_s) \tag{12}$$

and

$$\left[\frac{\omega^2}{K_0(\mathbf{r})} + \nabla\cdot\frac{1}{\rho_0(\mathbf{r})}\nabla\right]G_0(\mathbf{r},\mathbf{r}_s;\omega) = \delta(\mathbf{r}-\mathbf{r}_s), \tag{13}$$

where $G(\mathbf{r},\mathbf{r}_s;\omega)$ and $G_0(\mathbf{r},\mathbf{r}_s;\omega)$ are the free-space causal Green's functions describing wave propagation in the actual and reference media, respectively. The P-wave bulk modulus is $K = c^2\rho$, c is P-wave velocity, and ρ is the density. We assume both ρ_0 and c_0 are constants. For the simple 1D case, the perturbation V has the following form:

$$V(z,\nabla) = \frac{\omega^2\alpha(z)}{K_0} + \frac{1}{\rho_0}\beta(z)\frac{\partial^2}{\partial x^2} + \frac{1}{\rho_0}\frac{\partial}{\partial z}\beta(z)\frac{\partial}{\partial z}, \tag{14}$$

where $\alpha = 1 - K_0/K$ and $\beta = 1 - \rho_0/\rho$ are the two parameters we choose to perform the inversion.

Similar to equation 7, expanding V, α, and β in different orders of data and assuming the source and receiver depths are zero, we can determine the linear solution for α_1 and β_1 in the frequency domain (Zhang, 2006):

$$D(z,\theta) = -\frac{\rho_0}{4}\left(\frac{1}{\cos^2\theta}\alpha_1(z) + (1-\tan^2\theta)\beta_1(z)\right), \tag{15}$$

where $D(z,\theta)$ is a shot record $D(x,t)$ that is first Fourier-transformed over x and t to $D(k_g,\omega)$. Next, we perform a change of variables from temporal frequency to vertical wave number as $D(-2q_g,\theta)$ with $q_g = ((\omega/c_0)^2 - k_g^2)^{1/2}$ and $\tan\theta = k_g/q_g$, and finally it is inverse-transformed from $-2q_g$ to z to get $D(z,\theta)$. Please see equation 3.11 in Zhang (2006) for further details.

Let us consider the following logic. Equation 15 is an exact equation for the linear estimates $\alpha_1(z)$ and $\beta_1(z)$. Choosing two (or more) values of θ will represent the means to solve equation 15 for $\alpha_1(z)$ and $\beta_1(z)$.

For a single-reflector model, the left side of equation 15 is the migration of the surface-recorded data. The migration provides a step function at the depth of the reflector whose angle-dependent amplitude is the reflector's angle-dependent reflection coefficient.

The right side of equation 15 can be rewritten as

$$-\frac{\rho_0}{4}(\alpha_1(z)+\beta_1(z)+(\alpha_1(z)-\beta_1(z))\tan^2\theta). \quad (16)$$

Separately, we know that the exact plane-wave reflection coefficient is (e.g., Keys, 1989)

$$R(\theta)=\frac{(\rho_1/\rho_0)(c_1/c_0)\sqrt{1-\sin^2\theta}-\sqrt{1-(c_1^2/c_0^2)\sin^2\theta}}{(\rho_1/\rho_0)(c_1/c_0)\sqrt{1-\sin^2\theta}+\sqrt{1-(c_1^2/c_0^2)\sin^2\theta}}. \quad (17)$$

We can find a Taylor series in R as a function of $\sin^2\theta$ or another Taylor series using

$$\sin^2\theta=\frac{\tan^2\theta}{1+\tan^2\theta}. \quad (18)$$

This series is

$$\begin{aligned}R(\theta)&=R(\tan^2\theta)=R(\tan^2\theta=0)\\&+\left(\frac{dR(\tan^2\theta)}{d(\tan^2\theta)}\right)\bigg|_{\tan^2\theta=0}\cdot\tan^2\theta\\&+\left(\frac{dR^2(\tan^2\theta)}{d(\tan^2\theta)^2}\right)\bigg|_{\tan^2\theta=0}\cdot\frac{\tan^4\theta}{2}+\cdots\end{aligned} \quad (19)$$

Equation 19 is exact, and the amplitude of the step-function in equation 16 (after dropping the z-dependence) is

$$R(\tan^2\theta)=\alpha_1+\beta_1+(\alpha_1-\beta_1)\tan^2\theta. \quad (20)$$

The first term in the ISS is an exact equation for the linear estimates α_1 and β_1 of α and β, respectively.

Reconciling the exactness of equation 20 with the exactness of equation 19

Equation 20 would seem to represent a truncated, and therefore, approximate form of the Zoeppritz exact reflection coefficient (equation 19).

From the derivation of the inverse scattering series, equation 20 is not an approximation, but the exact equation for the linear estimates α_1 and β_1. On the other hand, equation 19 is the Zoeppritz equation and represents an indisputable cornerstone of elastic wave theory. The required consistency between equation 19 and 20 demands that α_1 and β_1 be functions of θ.

Let us see where that supposition then takes us from equation 20, which can be rewritten as:

$$R(\tan^2\theta)=\alpha_1(\theta)+\beta_1(\theta)+[\alpha_1(\theta)-\beta_1(\theta)]\tan^2\theta. \quad (21)$$

If two values of θ are chosen, say θ_1 and θ_2, then equation 21 will lead to two equations with four unknowns, $\alpha_1(\theta_1)$, $\alpha_1(\theta_2)$, $\beta_1(\theta_1)$, and $\beta_1(\theta_2)$. That is not good news. The problem here is that we have forgotten the basic meaning and starting point in defining α, β and α_1, β_1.

In a direct determination of a parameter from the ISS expansion in orders of the data, it is a critically important first step to ensure that the data (in terms of which a specific parameter is being expanded) are sufficient to determine that parameter. The data needed to determine a parameter are dependent upon what other parameters are (or are not) in the model. In other words, the required data is specified with the context in which that parameter resides (acoustic, elastic, and so forth).

Now consider a two-parameter world defined by $\alpha(z)$ and $\beta(z)$, and the expansions of α and β in orders of the data. In this case, if we suppose that α and β are expandable in terms of data at two different plane-wave angles, assuming that such a relationship between $D(z,\theta_1)$, $D(z,\theta_2)$ and α and β exists and is sufficient to determine α and β (not α_1 and β_1), then we can write the series for $\alpha(z)$ and $\beta(z)$ as

$$\begin{aligned}\alpha(z)&=\alpha_1(z,D(z,\theta_1),D(z,\theta_2))+\alpha_2(z,D(z,\theta_1),D(z,\theta_2))\\&+\cdots.\end{aligned} \quad (22)$$

In a compact notation,

$$\alpha(z)=\alpha_1(z,\theta_1,\theta_2)+\alpha_2(z,\theta_1,\theta_2)+\cdots, \quad (23)$$

where α_1 is the portion of α linear in the data set $(D(z,\theta_1),D(z,\theta_2))$. Similarly,

$$\beta(z)=\beta_1(z,\theta_1,\theta_2)+\beta_2(z,\theta_1,\theta_2)+\cdots. \quad (24)$$

If the model allowed only bulk modulus changes but not density variation, then the data required to solve for α would consist only of data at a single angle and in that single-parameter world,

$$\alpha(z)=\alpha_1(z,\theta_1)+\alpha_2(z,\theta_1)+\ldots. \quad (25)$$

Now in the two-parameter inverse problem, the data are

$$\begin{pmatrix}D(z,\theta_1)\\D(z,\theta_2)\end{pmatrix} \quad (26)$$

and then $D=G_0V_1G_0$ is equal to

$$\begin{pmatrix}D(z,\theta_1)\\D(z,\theta_2)\end{pmatrix}=\begin{pmatrix}(1+\tan^2\theta_1)&(1-\tan^2\theta_1)\\(1+\tan^2\theta_2)&(1-\tan^2\theta_2)\end{pmatrix}\times\begin{pmatrix}\alpha_1(z,\theta_1,\theta_2)\\\beta_1(z,\theta_1,\theta_2)\end{pmatrix} \quad (27)$$

and $\binom{\alpha_1(z,\theta_1,\theta_2)}{\beta_1(z,\theta_1,\theta_2)}$ is related linearly to $\binom{D(z,\theta_2)}{D(z,\theta_1)}$. The values of α_1 and β_1 will depend on which particular angles θ_1 and θ_2 were chosen, and that is anticipated and perfectly reasonable, because being a linear approximation in the data could (and should) be a different linear estimate depending on the data subset that is considered.

Equation 27 (a matrix equation) is the first term in the inverse series and determines α_1 and β_1, the linear estimate of α and β.

The key point

The lesson here is that the inverse problem does not start with $G_0V_1G_0=D$, but with $V=V_1+V_2+V_3+\ldots$ and the latter equation is driven by a view of which data set can determine the operator V.

This might seem like a somewhat useless academic exercise because equation 27 is the equation one would have solved for α_1 and β_1 if their θ dependence is ignored entirely. However, it is anything but academic. There are at least two problems with that conclusion.

The above analysis is valuable because (1) with α_1 and β_1 independent of θ, we have difficulty in claiming or satisfying the important requirement that the first equation in the inverse series is exact, and (2) more importantly, we can get into serious conceptual and practical problems in the elastic case if we do not have a very clear grasp of the underlying inverse issues and relationships in the acoustic case.

ELASTIC CASE

The scattering theory and the ISS for the 1D isotropic elastic earth are developed in Zhang and Weglein (2009a). We refer the reader to that paper (in this issue) for details of the elastic direct inverse and, in particular, for transforming the scattering equations from displacement to their PS representation.

In the displacement space

In the following, we start the inversion problem in two dimensions. The 2D elastic wave equation is (A. B. Weglein and R. H. Stolt, personal communication, 1992)

$$L\mathbf{u} \equiv \left[\rho\omega^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \partial_1 \gamma \partial_1 + \partial_2 \mu \partial_2 & \partial_1(\gamma - 2\mu)\partial_2 + \partial_2 \mu \partial_1 \\ \partial_2(\gamma - 2\mu)\partial_1 + \partial_1 \mu \partial_2 & \partial_2 \gamma \partial_2 + \partial_1 \mu \partial_1 \end{pmatrix} \right] \times \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbf{f}, \tag{28}$$

where $\mathbf{u} = \left[\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix} \right]$ = displacement, ρ = density, γ = bulk modulus ($\equiv \rho\alpha^2$ where α = P-wave velocity), μ = shear modulus ($\equiv \rho\beta^2$ where β = S-wave velocity), ω = temporal frequency (angular), ∂_1 and ∂_2 denote the derivative with respect to x and z, respectively, and $\mathbf{f}$ is the source term.

For constant $(\rho,\gamma,\mu) = (\rho_0,\gamma_0,\mu_0)$, $(\alpha,\beta) = (\alpha_0,\beta_0)$, the operator L becomes

$$L_0 \equiv \left[\rho_0\omega^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \gamma_0 \partial_1^2 + \mu_0 \partial_2^2 & (\gamma_0 - \mu_0)\partial_1 \partial_2 \\ (\gamma_0 - \mu_0)\partial_1 \partial_2 & \mu_0 \partial_1^2 + \gamma_0 \partial_2^2 \end{pmatrix} \right]. \tag{29}$$

Then for a 1D earth, defining $a_\rho \equiv \rho/\rho_0 - 1$, $a_\gamma \equiv \gamma/\gamma_0 - 1$ and $a_\mu \equiv \mu/\mu_0 - 1$ as the three parameters we choose for the elastic inversion, the perturbation $V = L_0 - L$ can be written as

$$V = -\rho_0 \begin{bmatrix} a_\rho\omega^2 + \alpha_0^2 a_\gamma \partial_1^2 + \beta_0^2 \partial_2 a_\mu \partial_2 & (\alpha_0^2 a_\gamma - 2\beta_0^2 a_\mu)\partial_1 \partial_2 + \beta_0^2 \partial_2 a_\mu \partial_1 \\ \partial_2(\alpha_0^2 a_\gamma - 2\beta_0^2 a_\mu)\partial_1 + \beta_0^2 a_\mu \partial_1 \partial_2 & a_\rho\omega^2 + \alpha_0^2 \partial_2 a_\gamma \partial_2 + \beta_0^2 a_\mu \partial_1^2 \end{bmatrix}. \tag{30}$$

For convenience (e.g., A. B. Weglein and R. H. Stolt, personal communication, 1992; Aki and Richards, 2002), we change the basis and transform the equations in the displacement domain to PS space, and finally, we do the elastic inversion in the PS domain.

Linear inversion of a 1D elastic medium in PS space

The equation for the first term in the ISS $D = G_0 V_1 G_0$ in the displacement domain can be written as the following form in the PS domain:

$$\begin{pmatrix} \hat{D}^{\mathrm{PP}} & \hat{D}^{\mathrm{PS}} \\ \hat{D}^{\mathrm{SP}} & \hat{D}^{\mathrm{SS}} \end{pmatrix} = \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} \begin{pmatrix} \hat{V}_1^{\mathrm{PP}} & \hat{V}_1^{\mathrm{PS}} \\ \hat{V}_1^{\mathrm{SP}} & \hat{V}_1^{\mathrm{SS}} \end{pmatrix} \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix}. \tag{31}$$

This leads to four equations:

$$\hat{D}^{\mathrm{PP}} = \hat{G}_0^P \hat{V}_1^{\mathrm{PP}} \hat{G}_0^P, \tag{32}$$

$$\hat{D}^{\mathrm{PS}} = \hat{G}_0^P \hat{V}_1^{\mathrm{PS}} \hat{G}_0^S, \tag{33}$$

$$\hat{D}^{\mathrm{SP}} = \hat{G}_0^S \hat{V}_1^{\mathrm{SP}} \hat{G}_0^P, \tag{34}$$

and

$$\hat{D}^{\mathrm{SS}} = \hat{G}_0^S \hat{V}_1^{\mathrm{SS}} \hat{G}_0^S. \tag{35}$$

For the P-wave incidence case (see Figure 1), assuming $z_s = z_g = 0$ and in the $(k_s,z_s;k_g,z_g;\omega)$ domain, the solution of equation 32 can be written as

$$\widetilde{D}^{\mathrm{PP}}(\nu_g,\theta) = -\frac{1}{4}(1 - \tan^2\theta)\widetilde{a}_\rho^{(1)}(-2\nu_g) - \frac{1}{4}(1 + \tan^2\theta)\widetilde{a}_\gamma^{(1)}(-2\nu_g) + \frac{2\beta_0^2 \sin^2\theta}{\alpha_0^2} \times \widetilde{a}_\mu^{(1)}(-2\nu_g), \tag{36}$$

where we used $k_g^2/\nu_g^2 = \tan^2\theta$ and $k_g^2/(\nu_g^2 + k_g^2) = \sin^2\theta$, and θ is the P-wave incident angle.

In the earlier section on acoustic inversion, β_0 and β_1 refer to relative changes in density, whereas in this elastic section β_0 and β_1 refer to relative change in shear-wave velocity. For the elastic inversion, in the special case when $\beta_0 = \beta_1 = 0$, equation 36 reduces to the acoustic two-parameter case equation 7 in Zhang and Weglein (2005) for $z_g = z_s = 0$:

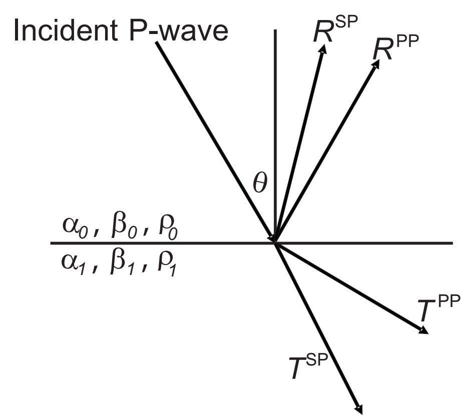

Figure 1. Response of incident compressional wave on a planar elastic interface. α_0, β_0, and ρ_0 are the compressional wave velocity, shear-wave velocity and density of the upper layer, respectively; α_1, β_1, and ρ_1 denote the compressional wave velocity, shear wave velocity, and density of the lower layer. The coefficients of the reflected compressional wave, reflected sheer wave, transmitted compressional wave, and transmitted shear wave are denoted by R^{PP}, R^{SP}, T^{PP}, and T^{SP}, respectively (Foster et al., 1997).

$$\widetilde{D}(q_g,\theta) = -\frac{\rho_0}{4}\left[\frac{1}{\cos^2\theta}\widetilde{\alpha}_1(-2q_g) + (1-\tan^2\theta) \times \widetilde{\beta}_1(-2q_g)\right]. \quad (37)$$

Direct nonlinear inversion of 1D elastic medium in PS space

The equation for the second term in the ISS $G_0V_2G_0 = -G_0V_1G_0V_1G_0$ in the displacement domain can be written in the PS domain as

$$\begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix}\begin{pmatrix} \hat{V}_2^{PP} & \hat{V}_2^{PS} \\ \hat{V}_2^{SP} & \hat{V}_2^{SS} \end{pmatrix}\begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} = -\begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} \times \begin{pmatrix} \hat{V}_1^{PP} & \hat{V}_1^{PS} \\ \hat{V}_1^{SP} & \hat{V}_1^{SS} \end{pmatrix}\begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix}\begin{pmatrix} \hat{V}_1^{PP} & \hat{V}_1^{PS} \\ \hat{V}_1^{SP} & \hat{V}_1^{SS} \end{pmatrix}\begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix}, \quad (38)$$

which leads to the four equations

$$\hat{G}_0^P\hat{V}_2^{PP}\hat{G}_0^P = -\hat{G}_0^P\hat{V}_1^{PP}\hat{G}_0^P\hat{V}_1^{PP}\hat{G}_0^P - \hat{G}_0^P\hat{V}_1^{PS}\hat{G}_0^S\hat{V}_1^{SP}\hat{G}_0^P, \quad (39)$$

$$\hat{G}_0^P\hat{V}_2^{PS}\hat{G}_0^S = -\hat{G}_0^P\hat{V}_1^{PP}\hat{G}_0^P\hat{V}_1^{PS}\hat{G}_0^S - \hat{G}_0^P\hat{V}_1^{PS}\hat{G}_0^S\hat{V}_1^{SS}\hat{G}_0^S, \quad (40)$$

$$\hat{G}_0^S\hat{V}_2^{SP}\hat{G}_0^P = -\hat{G}_0^S\hat{V}_1^{SP}\hat{G}_0^P\hat{V}_1^{PP}\hat{G}_0^P - \hat{G}_0^S\hat{V}_1^{SS}\hat{G}_0^S\hat{V}_1^{SP}\hat{G}_0^P, \quad (41)$$

and

$$\hat{G}_0^S\hat{V}_2^{SS}\hat{G}_0^S = -\hat{G}_0^S\hat{V}_1^{SP}\hat{G}_0^P\hat{V}_1^{PS}\hat{G}_0^S - \hat{G}_0^S\hat{V}_1^{SS}\hat{G}_0^S\hat{V}_1^{SS}\hat{G}_0^S. \quad (42)$$

Because $\hat{V}_1^{PP}$ relates to $\hat{D}^{PP}$, $\hat{V}_1^{PS}$ relates to $\hat{D}^{PS}$, and so on, the four components of the data will be coupled in the nonlinear elastic inversion. Therefore, we cannot perform the direct nonlinear inversion without knowing all components of the data. Equations 31–42 represent the necessary and sufficient data requirements for the linear and higher-order direct inversion for any one of the elastic mechanical property changes. Each of the linear and higher-order terms is the unique expansion of that mechanical property in terms of a data that can invert directly for those quantities.

The three parameters we seek to determine are

- $a_\gamma \rightarrow$ relative change in bulk modulus
- $a_\rho \rightarrow$ relative change in density
- $a_\mu \rightarrow$ relative change in shear modulus

These parameters are to be expanded as a series in the data. Which data?

The answer is once again the data needed to directly determine those three quantities.

The thesis of Zhang (2006) demonstrates for the first time not only an explicit and direct set of equations for improving upon linear estimates of the changes in those elastic properties, but also perhaps equally and even more importantly, the absolutely clear data requirements for determining a_γ, a_ρ, and a_μ.

The data requirements are

$$D = \begin{pmatrix} \hat{D}^{PP} & \hat{D}^{PS} \\ \hat{D}^{SP} & \hat{D}^{SS} \end{pmatrix} \quad (43)$$

for a 2D earth and generalize to a 3×3 matrix for a 3D earth with SH and SV shear waves.

The 2D message is delivered in equation 38 (or equations 39–42) that the first nonlinear contribution to a_γ, a_μ, and a_ρ requires that data; and hence, the exact determination of those elastic quantities also requires that data set (Weglein, 2009):

$$\begin{pmatrix} V^{PP} & V^{PS} \\ V^{SP} & V^{SS} \end{pmatrix} = \begin{pmatrix} V_1^{PP} & V_1^{PS} \\ V_1^{SP} & V_1^{SS} \end{pmatrix} + \begin{pmatrix} V_2^{PP} & V_2^{PS} \\ V_2^{SP} & V_2^{SS} \end{pmatrix} + \cdots. \quad (44)$$

The logic is as follows:

$$\begin{pmatrix} a_\gamma \\ a_\mu \\ a_\rho \end{pmatrix}$$

requires

$$\begin{pmatrix} \hat{D}^{PP} & \hat{D}^{PS} \\ \hat{D}^{SP} & \hat{D}^{SS} \end{pmatrix}$$

because

$$\begin{pmatrix} a_\gamma^{(2)} \\ a_\mu^{(2)} \\ a_\rho^{(2)} \end{pmatrix}$$

requires

$$\begin{pmatrix} \hat{D}^{PP} & \hat{D}^{PS} \\ \hat{D}^{SP} & \hat{D}^{SS} \end{pmatrix}.$$

Hence

$$\begin{pmatrix} a_\gamma^{(1)} \\ a_\mu^{(1)} \\ a_\rho^{(1)} \end{pmatrix}$$

must mean linear in

$$\begin{pmatrix} \hat{D}^{PP} & \hat{D}^{PS} \\ \hat{D}^{SP} & \hat{D}^{SS} \end{pmatrix},$$

i.e., linear in the data needed to determine

$$\begin{pmatrix} a_\gamma \\ a_\mu \\ a_\rho \end{pmatrix}.$$

Inverting

$$\hat{D}^{\mathrm{PP}} = G_0^P V_1^{\mathrm{PP}} G_0^P \quad (45)$$

alone for $a_\gamma^{(1)}$, $a_\mu^{(1)}$, and $a_\rho^{(1)}$, although mathematically achievable, corresponds to a linear approximate forward problem for PP data solved in an "inverse sense." The direct inversion of the elastic heterogeneous wave equation defines the data needed to invert for those quantities, in principle. A linear inversion is a first and linear approximate term in a series solution that inverts data directly from the elastic wave equation for changes in earth's mechanical properties.

That definitive, linear inverse definition requires the linear approximation to be linear in the collection of data components. In the simpler case of acoustics, and inverting the heterogeneous acoustic equation directly for changes in acoustic properties, the direct inverse solution (Zhang, 2006) provided by the ISS, is a series in the measured pressure wavefield, and the linear acoustic inverse is linear in the collection of measurements of the pressure wavefield needed to solve the direct inverse. But the linear inverse in the elastic case is linear in all of the data components, because the direct elastic inversion is an expansion in all data components.

Solving for $a_\gamma^{(1)}$, $a_\mu^{(1)}$, and $a_\rho^{(1)}$ from $\hat{D}^{\mathrm{PP}}$ alone is a model matching of PP data and something less than a linear inverse.

The ISS and task-specific subseries first need to treat the linear term with respect and then the higher-order terms can carry out their purpose.

If the linear estimate is not calculated correctly, the ISS cannot recover or compensate — it wants the linear estimate to be the linear estimate, and never expects it to be exact or close to exact, but it never expects it to be less than linear. Let linear be linear.

The power and promise of the ISS derives from its deliberate, direct, physically consistent, and explicit nature. It recognizes that when there is any perturbation in a medium, the associated perturbation in the wavefield always is related nonlinearly to that change.

The inverse implies that the medium perturbation itself is related nonlinearly to the perturbation in the wavefield. Thus, the medium-property perturbation operator is related nonlinearly to the change in the wavefield on the measurement surface, i.e., to the measured data.

We assume the scattered field and the perturbation can be expanded in orders of the medium perturbation V and the measured data D, respectively:

$$\psi_s = (\psi_s)_1 + (\psi_s)_2 + (\psi_s)_3 + \dots \quad (46)$$

and

$$V = V_1 + V_2 + V_3 + \cdots, \quad (47)$$

where $(\psi_s)_n$ is the portion of ψ_s which is the nth order in V and where V_n is the portion of V which is the nth order in the data D, i.e., the measured values of ψ_s. The entire foundation behind the ISS is based on equations 46 and 47, expressing the indisputable nonlinear relationship between changes in medium properties and the concomitant changes in wavefields. This is all that needs to be assumed. These equations simply communicate the identity known as the Lippmann-Schwinger equation, which governs perturbation theory, and its forward, nonlinear modeling series and nonlinear inverse-series forms.

Beyond that point, the process and procedure for determining $V_1, V_2, V_3, \dots$ is out of our hands and away from our control. How to find V_1 from D is prescribed and what to do with V_1 to determine V_2 is prescribed also. That nonlinear explicit and direct nature, and the steps to determine those terms $V_1, V_2, V_3, \dots$ are not decision-making opportunities. If we decide what to do with V_1 rather than have the nonlinear relationship between data and V decide, then we step away from a single and defined physics into, e.g., the math world of iterative linear inversion or model matching. How do we formulate a multiple-removal algorithm concept in an iterative linear inverse or model-matching scheme? How do we formulate a model-type independent multiple-removal method from a full-waveform inversion, or any indirect-inversion model-matching procedure? The latter aims immediately to either improve or match the model properties with the subsurface. From an inverse-scattering-series perspective, the latter all-or-nothing strategy (1) misses the opportunity to achieve other useful but less daunting tasks such as multiple removal and depth imaging; and (2) begins at the first step straight into the most challenging task of parameter estimation, with all the pitfalls of insufficient model types and bandwidth sensitivities.

For the ISS, the decisions are not under our control or influence. It has one physical reference model, the water, and a single *unchanged* separation of the earth into a reference medium and a general perturbation operator that can accommodate a very wide range of earth model types. The model types need not be specified unless we want the direct, nonlinear AVO subseries. The ISS provides a set of direct equations to solve, with an analytic, unchanged inverse operation.

The physics-consistent direct-inverse formalism of the inverse scattering series stands alone in predicting that we require all four components of the data $\begin{pmatrix} \hat{D}^{\mathrm{PP}} & \hat{D}^{\mathrm{PS}} \\ \hat{D}^{\mathrm{SP}} & \hat{D}^{\mathrm{SS}} \end{pmatrix}$ to even estimate elastic properties linearly. Iterative linear inversion tries to substitute a set of constantly changed, forward problems with linear updates for a single, entirely prescriptive, consistent, and explicit nonlinear physics. The latter is the inverse scattering series; the former (iterative linear inversion) has an attraction to linear inverses (and generalized inverses), which have no single physical theory and consistency. Linear inversion and generalized inverse theory are part of standard graduate training in geophysics; hence it is easy to understand trying to recast the actual nonlinear problem into a set of iterative linear problems where the tools are familiar. Model-matching schemes and iteratively linear inversion are reasonable and sometimes useful but they are more math than physics. Thus, they have no way to provide the framework for inversion that equations 46 and 47 provide by staying consistent with physics.

The practical, added value that direct ISS nonlinear inversion provides beyond linear inversion is described in Zhang (2006), and Zhang and Weglein (2005, 2006, 2009a, and 2009b). There are circumstances in which very different target lithologies have very similar changes in mechanical properties. The added value is demonstrated in 4D application in discriminating between pressure and fluid-saturation effects. That distinction results in the difference between a drill and a no-drill decision.

DISCUSSION

Indirect inverse methods (e.g., model matching, cost-function search engines, optimal stacking, full-waveform inversion, and iterative linear inversion) at best seek to emulate or to satisfy some property or quality of an inverse solution, rather than providing the solution directly. Here we communicate a message on the critical distinction that is often ignored between modeling and inversion, and the even greater difference between direct-inverse solutions and indirect methods that seek that same goal.

We describe the algorithmic and practical consequences of this increased conceptual clarity. In particular, we examine the commonly held view that considers PP reflection data (e.g., Stolt and Weglein, 1985; Boyse and Keller, 1986) to be adequate for estimating changes in mechanical properties, and that is used today in methods for both linear and nonlinear estimates of mechanical property changes across a reflector. We show, from the definitiveness of a direct-inversion perspective, that PP data are fundamentally insufficient. The direct inversion for those changes in mechanical properties provided by the ISS communicates that all components of data (PP, PS, SS,…) are required for either linear and/or nonlinear direct inversion. Linear inversion is defined here as the linear approximate solution to the direct inverse problem. For indirect methods or methods with modeling as a starting point, there is no reason to suspect or conclude that PP data would be fundamentally and conceptually inadequate. Indirect methods are neither equivalent to nor a substitute for direct methods. We point out the general conceptual and algorithmic differences.

The direct nonlinear solution given by the ISS provides the first unambiguous and consistent meaning for a direct, approximate linear inverse solution. Inverting PP data linearly for approximate changes in earth's mechanical properties provides a linear approximate solution to the PP data equation, but not a linear approximate inverse solution for changes in earth's mechanical properties. To achieve the higher bar of a linear approximate inverse solution requires a nonapproximate inverse solution, as a starting point, as either a closed form or expressed as a series that is going to be reduced and simplified in a linear approximate form. The ISS represents a nonapproximate fully nonlinear and direct inverse solution. The direct inversion of earth's mechanical properties requires PP, PS, and SS data in a 2D world, and PP, PSv, PSh, SvSv, ShSh, and SvSh in a 3D earth. Hence, the linear approximate inverse solution must be linear in the data that allow the linear solution to correspond to the linear approximation of the inverse solution. The PP data alone can produce an approximate solution to a forward PP equation, but PP, PS, and SS can provide a linear approximate inverse solution.

Hence, the conclusion is that only multicomponent data can produce a linear approximate inversion solution, which is the first step toward a complete nonlinear and direct solution.

We recognize that the changes in material properties across a single reflector and the corresponding reflection coefficients and reflection data have a nonlinear relationship in a modeling and therefore an inversion sense. However, the key point is that although changes in earth's mechanical properties at an interface can (through the Zoeppritz relations) directly, nonlinearly, exactly, and separately determine each of the PP, PS, and SS reflection coefficients, it requires all of those reflection coefficients taken together to determine any one or more changes in mechanical properties. That message is neither obvious nor reasonable, or even plausible. However, the message here is that it is all of those difficult and unattractive things, and yet it is also unambiguously and unmistakably true. In general, inversion or processing is not modeling run backward.

Direct linear and indirect methods (e.g., full-waveform inversion) have not and cannot bring that clarity to the meaning and unambiguous prescription of the linear approximate inverse solution. Model matching with global searches of PP data alone have no framework or other reason to suspect the fundamental inadequacy of that PP data to provide a linear inverse, let alone a nonlinear solution. We have published using PP data to estimate changes in physical properties, and along with the entire petroleum industry, we have used PP data in AVO analysis. The PP data have enough degrees of freedom, given enough angles, to more than solve for linear estimates in changes in earth's material properties. So what is the problem?

We are fully aware that a single angle of data cannot invert simultaneously for several changes in earth's mechanical properties because the degrees of freedom in the data need to be the same as in the sought-after earth's material properties. This is recognized and understood in inverse theory. Sufficient degrees of freedom in your data are a necessary but not a sufficient condition for a linear inverse solution, although it is necessary and sufficient for solving a direct-forward-PP relationship in an inverse sense. The fact that all components of elastic data are absolutely baseline required to provide a meaningful linear inverse or nonlinear inverse solution is a new, clearer, and higher bar, and a much more subtle, but in no way less-important message. The fact that the ISS is the only direct and nonlinear inversion method has allowed it to:

1) Stand alone and provide a framework for the very meaning of linear inverse.
2) Provide a systematic and precise way to improve upon those estimates directly through higher terms in the expansion of those earth's mechanical properties directly in terms of the data. The required data are full multicomponent data and not only PP.

If we have an expansion for a change in a physical property (call it V, in terms of reflection data D) then schematically, $V(D) = V(D = 0) + V'(D = 0)D + \frac{1}{2}V''(D = 0)D^2 + \cdots$, where $V(D = 0) = 0$, $V'(0)D$ is the linear estimate to $V(D)$, and D are the data needed to determine $V(D)$. Only the ISS provides the precise series for $V(D)$ and, hence, in that process defines both the data necessary to find $V(D)$ and its linear estimate $V_1 = V'(0)D$. We cannot change the expansion variable in a Taylor series. If the data D determine the series, then each term including the first linear term depends on all elements of D. The data D are multicomponent data for the determination of changes in elastic properties. That is the point.

The need for multicomponent data does not add a set of constraints beyond PP data, but provides the necessary baseline data needed to satisfy the fundamental nonlinear relationship between reflection data and changes in earth's mechanical properties. It is a fundamental data need that stands with data dimensionality and degrees of freedom. It comes in at the ground floor, before more subtle and important issues of robustness and stability are examined — it is not merely a practical enhancement or boost to PP-data inversion potential and capability. The need for multicomponent data is fundamental. As with other things, it can be ignored but rarely will be ignorable.

The latter PP data are fundamentally inadequate from a conceptual and math-physics analysis perspective for a consistent and meaningful target identification, and the needed data and methods for using that data are provided only by the directness and fully nonlinear and prescriptive nature of the ISS. Those unique properties and benefits of the ISS are not provided by either (1) linear approximate direct-inverse methods, behind all current mainstream leading-edge migration and migration-inversion algorithms, or (2) nonlinear indirect inverse methods such as iterative linear or other indirect model-matching inversion methods, or full-waveform inversion.

We have taken the reader through the thinking process and deliberation within our group that brought this issue to light. It began in the simpler acoustic world, where the difference between the forward and inverse problem needed some attention and clarification. We have raised and answered the following questions:

1) What does linear in the data mean?
2) Linear in what data? What are the actual data requirements

needed to define a linear inverse as a linear approximation "in data" to the solution of the direct nonlinear inverse problem?

3) Conservation of dimension (having enough degrees of freedom in the data to "solve" an equation) is not a sufficient condition to define "what data," and being able to solve an equation (in isolation) is not the same as finding a physically meaningful solution or even a linear estimate.
4) Solving an equation without the context and framework within which that equation resides, and ignoring the assumptions that lead to that equation, constitutes a dangerous and ill-considered path.
5) What are the implications for data collection and target identification?

In summary, (1) PP data are necessary and sufficient for a direct inversion of an acoustic medium/target, and hence PP is necessary and sufficient for a linear inversion for acoustic properties, but (2) all components of data PP, PS, SS,… are necessary and sufficient for a direct inversion of an elastic medium/target (provided explicitly in Zhang, 2006, pp. 77). Hence all components are required for a linear approximate inversion for elastic properties. The linear inverse is the first and linear approximation of those parameters in a series that is a nonlinear expansion in terms of data that, in principle, can determine those properties directly.

Zhang (2006, p. 73–75) asks and answers this question, mentioned in item two in the list above: What is one to do for direct nonlinear AVO of an elastic medium/target when one measures only PP data, as in typical towed-streamer marine data within the water column?

The response was to use the PP data in a forward-PP relationship and solve that in a traditional manner with three (or more angles) for three parameters, and then use two of those three parameters to synthesize the required PS, SS… components necessary to compute direct nonlinear inversion of the elastic properties, which is better than putting zeros in places where the direct inversion expected PS, SS… data. This is the same issue that Matson (2000) faces in the direct elastic inverse scattering series for ocean-bottom and onshore-multiple removal. The need for multicomponent data arises as an absolutely necessary requirement for a direct elastic inversion for AVO purposes or for the direct removal of multiples when the measurement surface is the ocean bottom or onshore (land) and requires an elastic reference medium.

An important point here is that the synthesized PS, SS,…and the actual PS, SS,… data never are equal (see Zhang, 2006, pp. 73–75, for several examples). The inability to use PP data alone to produce the same linear inversion as having PP, PS, and SS data is noteworthy. That inability would not be the case if a linear inverse of PP data could produce the other data components, then inverting either PP alone or PP, PS, and SS together would make no difference. It makes a difference, and it supports the inverse-scattering-series message that PP data is, in principle, inadequate to directly invert for changes in the mechanical properties of the earth. This illustrates and highlights the distinction and message that our study conveys for AVO applications. For imaging, the *indirect* methods such as common image gather, CFP, CRS, and optimal moveout trajectory stacking ("path integral"), all have surrogates and proxies for a velocity model, and yet sometimes portray the proxy as though it was somehow beyond, above, or independent of velocity. In fact it is an attempt and weak (necessary but not sufficient) substitute for, and admission that velocity is what they seek, but the velocity is beyond their reach. All of these indirect methods believe that a *direct* depth imaging method would require an accurate velocity. The only multi-D *direct* inversion, the inverse scattering series, stands alone in its message that a direct depth imaging method derives from the ISS without a velocity model.

The role of direct and indirect methods

At this point, we feel it is important to mention that a clear and important role exists for indirect methods, which we recognize and appreciate. Among authors who recognize the need for a judicious use of direct and indirect methods are: Verschuur et al., 1992; Carvalho and Weglein, 1994; Berkhout and Verschuur, 1995; Matson, 2000; Abma et al., 2005; Weglein and Dragoset, 2005; Kaplan and Innanen, 2008. Indirect methods always are needed to complement and fill the gap between our deterministic direct methods and the complexity of the actual seismic experiment, the real subsurface, and the realities and compromises of acquisition. Adaptive methods are called upon, and useful, and the part of reality outside our modeled physics needs serious attention as well. Treating the seismic inverse problem as entirely direct inversion, or (as more often is the case) entirely indirect, does not recognize or benefit from the mix of distinct issues they address, and from pooling their necessary strengths for field data application. However, in some general and overriding sense, overall scientific and practical progress is measured as the boundary between the two moves to bring more issues within the sphere of physics, and addressable by direct deterministic tools and away from the computational world of search engines (full wavefield or otherwise) and error surfaces.

Finally, we note that the first and linear term of the elastic inverse problem was influenced not only by the nonlinear term; in fact, it was defined by that term. That data-requirement message, along with the entire inverse-series apparatus, results from the observation that the perturbed wavefield and the concomitant medium perturbation are related nonlinearly. Honor and respect that fundamental nonlinear relationship and a physics-driven set of direct, consistent, deliberate, and purposeful inversion algorithms, and a clear platform and unambiguous framework (that explains earlier anecdotal experiences) are the dividend and value.

CONCLUSION

A unique and unambiguous data-requirement message is sent from the inverse scattering series for linear and nonlinear direct inversion. Other methods and approaches look at the inverse problem, e.g., either linear or beyond linear, but iterative linear or model-matching indirect inversion methods, including so-called full-wavefield inversion, never have and never will provide that clarity and definition. Nothing other than a direct inversion ought to provide confidence that we are solving the problem in which we are interested. The inverse scattering series defines the data and algorithms needed to carry out direct nonlinear inversion. That is the starting point for defining a linear inverse approximate solution. A linear inverse solution is a linear approximation to the inverse solution. A linear estimate of parameters determined using a relationship between those parameters and any convenient data, typically from a forward or modeling relationship, does not warrant being labeled a linear approximation to the inverse solution. That is the essential point. Linear should mean linear with respect to the data adequate to determine the actual inverse solution.

How do we know which data are adequate? Looking at modeling equations is the wrong starting point for understanding inversion, and the proof is that looking at modeling PP data as a starting point seems reasonable and plausible, but it is fundamentally wrong for looking at the starting point and guide for the inverse solution and linear inverse estimates therein. The inverse problem is not the forward problem run backward. Legitimate inverse solutions do not begin with taking a forward solution and trying to solve that relationship in an inverse sense for changes in medium properties that occur in the forward relationship. This is the crux of the logical flaw in all current AVO, full-waveform inversion, and indirect methods. It is an essential point for clear understanding of the foundation behind our processing algorithms and for the design and effective use of target identification and parameter estimation methods.

Modeling and forward predicting, and creating multiples by any modeling method, (e.g., finite difference or the forward scattering series) require precise and detailed subsurface information about everything in the subsurface the multiple has experienced. However, the inverse scattering series has distinct subseries for removing free-surface and internal multiples that provide algorithms which require absolutely no subsurface information, and are the same algorithms for acoustic, elastic, anisotropic, and anelastic media. Not one line of code changes if the earth is acoustic, elastic, anisotropic, or anelastic. That is amazing, and it points out very clearly the flaw in thinking of inversion as starting with a modeling idea or formula and then treating inversion as a form of model matching, or forward modeling run backward. How could one even imagine model matching and subtracting multiples independent of the type of earth one is adopting and modeling?

A recent and dominant trend in many fields of inversion, including seismic inversion, is to ignore the two kinds of inversion, direct and indirect, and even go so far as to define inversion as indirect model matching, or "full-waveform" with a big computer. This study shows certain pitfalls and serious dangers of using indirect methods. It provides a necessary and timely reminder of the two types of inversion and the unique strengths, clarity, guidance, and understanding that direct inversion represents.

We can model match D_{pp} or iteratively invert D_{pp} until the cows come home (i.e., ad infinitum), and we will find ambiguities and resolution challenges. When those methods use more components of data, they sometimes produce less ambiguity and better resolution, but from, e.g., a model-matching or full-waveform-inversion perspective, one never guesses why. The iterative linear inverse of PP data is nonlinear in PP data, but it is not a nonlinear direct inverse solution because it does not recognize that all components PP, PS, SS,… are needed and hence has no chance of agreeing with the direct nonlinear inverse provided only by the inverse series.

In a separate issue, the minimally realistic earth model for amplitude analysis is an elastic medium that generates elastic wavefield data and is characterized by elastic reflection coefficients. It is an issue of serious conceptual and practical concern to use an acoustic inverse, especially when using amplitude analysis, for synthetic or field data generated by an elastic medium. Much of current inversion practice and methodology uses the wrong data, an unrealistic earth-model type, and algorithms mislabeled as inversion.

We have presented a new and previously unrecognized and unheralded benefit of the fully nonlinear and direct multidimensional inversion represented by the ISS. That new contribution is at the core of all inversion theory. It impacts how we better understand previously observed and reported results from different groups and researchers, and it provides a firm, unambiguous platform and guide to researchers and explorationists. It allows us to understand, for the very first time, the data collection mandated and required for a meaningful and consistent linear approximate inverse solution. In addition, it gives us a direct prescription and determination of the linear estimate and a framework and systematic methodology for nonlinear target identification.

ACKNOWLEDGMENTS

We thank all sponsors of the Mission-Oriented Seismic Research Program (M-OSRP) for their support and encouragement. We have been partially funded by and are grateful for the National Science Foundation and Collaboration between Mathematics and Geosciences (NSF-CMG) award DMS-0327778 and the Department of Energy (DOE) Basic Sciences award DE-FG02-05ER15697. We thank ConocoPhillips, WesternGeco, and PETROBRAS for permission to publish. We express our deepest appreciation to Jingfeng Zhang for sharing his valuable insights, his careful and detailed analysis, and his unique and profound perspective. A. B. Weglein would like to thank R. H. Stolt, Doug Foster, Bob Keys, Tadeusz Ulrych, and Joe Keller for stimulating, interesting, and engaging discussions.

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